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The symmetry of the CS condition on one-sided ideals in a prime ring

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Abstract

Let R be a prime ring and $e \in R$ be an idempotent. We show that eR_R is nonsingular, CS and $1 < \text{u-dim}(eR_R) < \infty$ if and only if ${}_RRe$ is nonsingular, CS and $1 < \text{u-dim}({}_RRe) < \infty$.

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1. Introduction

Throughout this note we consider associative rings with identity. All modules are unitary modules. For a module M over a ring R we write M_R (${}_RM$) to indicate that M is a right (left) R-module. The uniform dimension of a module M is denoted by u-dim(M). For all notation for modules and rings not defined here we refer the reader to the texts [1,5,10], and [11].

A module M is called a CS (or extending) module if every submodule of M is essential in a direct summand of M.

A ring R is called a right CS ring if the right R-module R_R is CS. Similarly we can define left CS rings. For basic background on CS modules and CS rings we refer the reader to the text [4].

In [9] (see also [6, Theorem 12.8B]) a (left–right) symmetry result for prime rings is obtained which can be presented as follows:

Theorem 1. *For a prime ring R the following conditions are equivalent:*

(a) *R is right Goldie, right CS and* $\text{u-dim}(R_R) > 1$*;*
(b) *R is left Goldie, left CS and* $\text{u-dim}({}_RR) > 1$.

As noticed in [9], this theorem is not true in general if we remove the condition $\text{u-dim}(R_R) > 1$ or $\text{u-dim}({}_RR) > 1$ in (a) or (b), respectively.

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Let e be an idempotent in a prime ring R. From Theorem 1 it is natural to ask the question of under what conditions the CS property of eR_R implies that of ${}_RRe$. Besides Theorem 1, our question is motivated by the known and useful fact that for a semiprime ring R and an idempotent $e \in R$, eR_R is semisimple if and only if ${}_RRe$ is semisimple if and only if eRe is a semisimple artinian ring.

In this note, we apply Theorem 1 to study this question and prove a theorem that provides a symmetry result in a more general setting where the prime ring R may have infinite uniform dimension (see Theorem 3).

2. The result

Let N be a submodule of a module M_R where R is a ring. N is said to be an essential submodule of M if for any nonzero submodule C of M, $C \cap N \neq 0$. A submodule U is uniform if every nonzero submodule of U is essential in U. If $N \subseteq N'$ are submodules of M such that N is essential in N', then we say that N' is an essential extension of N in M. By Zorn's Lemma, every submodule N of M has a maximal essential extension N^* in M, and N^* is said to be the (essential) closure of N in M. A submodule $C \subseteq M$ is called a closed submodule if C is the maximal essential extension of itself in M.

The set Z of the elements of M that are annihilated by an essential right ideal of R is a well-defined submodule of M. This submodule Z is called the singular submodule of M. Moreover, Z is a fully invariant submodule of M; this means that for any R-endomorphism $\varphi : M \to M$, $\varphi(Z) \subseteq Z$. If $Z = 0$ then M is called a nonsingular module; M is called a singular module if $M = Z$. A useful fact about nonsingular modules is that the closure of any submodule of a nonsingular module is unique.

A ring R is called right (left) nonsingular if the right (left) R-module R is nonsingular, or in other words, if for any essential right (left) ideal $E \subseteq R$ and $r \in R$, $rE = 0$ ($Er = 0$) implies $r = 0$. Since we consider only (associative) rings with identity, our rings are never singular. For more information on singular and nonsingular modules we refer the reader to the texts [4] and [8].

As described in [11], for any right nonsingular ring R, its maximal right quotient ring $Q^r_{\max}(R)$ is obtained as a ring which is isomorphic to the endomorphism ring of the injective hull $E = E(R_R)$ of the module R_R (cf. [11, XII. 2.3]). But, as R_R is right nonsingular, the multiplication in R can be extended in a unique way to E making E have a ring structure such that E is an R-bimodule. Therefore $End_R(E) \cong E(R)$ as rings, and so we may identify E as the maximal right quotient ring of R. Similarly, the injective hull of ${}_RR$ is the maximal left quotient ring $Q^l_{\max}(R)$ of R provided R is left nonsingular. In general these two quotient rings are different. The following result by Chase and Faith [2] (see also [11, XII. 2.4]) plays a crucial role in our investigations.

Lemma 2. *For a right nonsingular ring R the following conditions are equivalent:*

(i) *$Q^r_{\max}(R)$ is a direct product of endomorphism rings of vector spaces over division rings.*
(ii) *$Q^r_{\max}(R)$ is an essential extension of its right socle.*
(iii) *Every nonzero right ideal of R contains a uniform right ideal.*

In general, the right nonsingularity of a ring does not imply the same property on the left. But it will do if we assume additionally that the ring is prime and contains a nonsingular CS right ideal of finite uniform dimension greater than 1. We will see this in the proof of Theorem 3 below.

Theorem 3. *Let R be a prime ring with $e = e^2 \in R$. Then the following conditions are equivalent:*

(i) *eR_R is nonsingular, CS and $1 < \text{u-dim}(eR_R) < \infty$;*
(ii) *${}_RRe$ is nonsingular, CS and $1 < \text{u-dim}({}_RRe) < \infty$;*
(iii) *eRe is right Goldie, right CS and $\text{u-dim}(eRe_{eRe}) > 1$;*
(iv) *eRe is left Goldie, left CS and $\text{u-dim}({}_{eRe}eRe) > 1$.*

Proof. (i) $\Rightarrow$ (iii). We divide the proof of this implication into several claims. Let us denote by $Q^r_{\max}$ ($Q^l_{\max}$) the maximal right (left) quotient ring of R.

Claim 1. $e = e_1 + \cdots + e_n$ where each e_iR is uniform and $\{e_i\}_{i=1}^n$ is a set of orthogonal idempotents.

This is folklore, but we prove it for the sake of completeness. Since eR is CS and of finite uniform dimension, $eR = U_1 \oplus \cdots \oplus U_n$ where each U_i is a uniform submodule of eR. Write $e = e_1 + \cdots + e_n, e_i \in U_i$. Then $e_i = ee_i = e_1e_i + e_2e_i + \cdots + e_ie_i + \cdots + e_ne_i$. On the other hand, e_i has the representation $e_i = 0 + 0 + \cdots + e_i + \cdots + 0$.

Since the representation of e_i as a sum of elements from U_j is unique, we may compare these two equations of e_i to get $e_i^2 = e_i$, $e_j e_i = 0$ for $i \neq j$ where $i, j \in \{1, 2, \dots, n\}$. Hence $eR = e_1 R \oplus \cdots \oplus e_n R$. Notice that, as $n > 1$, the uniform dimension of eRe_{eRe} is greater than 1.

Claim 2. R is right nonsingular.

Let Z be the right singular ideal of R. If $Z \neq 0$, then, as R is a prime ring, $eR \cap Z \neq 0$. However this means that eR is not nonsingular, a contradiction. Thus $Z = 0$, i.e., R is right nonsingular, completing the proof of Claim 2.

Let U be the sum of all uniform right ideals of R. Then in the light of (i), U is nonzero. Let $0 \neq r \in R$, and V be a uniform right ideal of R. Then the map $f : x \mapsto rx, \forall x \in V$ is a homomorphism $V \to rV$. Since R is right nonsingular, $ker(f) = 0$, i.e., $V \cong rV$, in particular, rV is a uniform right ideal of R. Using this fact we see that U is a two-sided ideal of R. As R is prime, U must be essential in R_R. This implies that every nonzero right ideal of R contains a uniform right ideal. By Lemma 2, $Q^r_{\max}$ is the direct product of minimal right ideals. Since each $e_i R$ is uniform, we see that each $e_i Q^r_{\max}$ is a minimal right ideal of $Q^r_{\max}$. Note that, as described before Lemma 2, $(Q^r_{\max})_R$ is the injective hull of R_R. Hence $eQ^r_{\max} = e_1 Q^r_{\max} \oplus \cdots \oplus e_n Q^r_{\max}$.

To complete the proof of this part we verify the following claim:

Claim 3. eRe is a right Goldie, right CS ring.

As just described, $eQ^r_{\max}$ is in fact a semisimple right ideal of $Q^r_{\max}$. Because $End(eQ^r_{\max}) \cong eQ^r_{\max}e$, we see that $eQ^r_{\max}e$ is a semisimple artinian ring which is the classical right quotient ring of eRe. In particular, eRe is a right Goldie ring. Since $eRe \cong End(eR)$ it is clear that u-dim$(eRe_{eRe}) > 1$ (see also a notice before Claim 2). Now we show that eRe is a right CS ring.

Let V be a closed uniform right ideal of eRe. Clearly V_{eRe} is essential in a minimal right ideal $f(eQ^r_{\max}e)$ of $eQ^r_{\max}e$ with $f = f^2 \in eQ^r_{\max}e$. Since $f = fe = ef$, $f(eQ^r_{\max}e) = fQ^r_{\max}e$. It follows that $V \subseteq fQ^r_{\max}$. Moreover, as $fQ^r_{\max}f$ is a division ring, $fQ^r_{\max}$ is a minimal right ideal of $Q^r_{\max}$. Consequently $K = fQ^r_{\max} \cap eR$ is a closed uniform submodule of eR that contains V. By (i), $eR = K \oplus H$ for some submodule H of eR, i.e., $K = g(eR)$ for some idempotent $g \in eR$. On the other hand, as eR (and R) is nonsingular and $fQ^r_{\max}$ is the injective hull of K, we must have $gQ^r_{\max} \subseteq fQ^r_{\max}$. Hence $gQ^r_{\max} = fQ^r_{\max}$ by the minimality of $fQ^r_{\max}$. Consequently, V is essential in $g(eRe) \subseteq f(eRe)$; therefore $V = g(eRe)$. Set $t = ge$ ($=ege \in eRe$). Then $t^2 = (eg)(eg) = e(g^2)e = ege = t$ ($\in eRe$). Thus $V = g(eRe) = (ge)(eRe) = t(eRe)$, and so V is a direct summand of eRe_{eRe}. Since eRe has finite right uniform dimension, we can inductively prove that every closed right ideal of eRe is a direct summand; hence the proof of Claim 3, and consequently that of (i) $\Rightarrow$ (iii), is complete.

(iii) $\Rightarrow$ (i). Let E be an essential right ideal of R; then clearly eE is an essential submodule of eR. If there is a nonzero right ideal L of eRe such that $L \cap eEe = 0$, then for the closure L^* of L in eRe also $L^* \cap eEe = 0$. But eRe is right CS; hence there is an idempotent $g \in eRe$ with $L^* = g(eRe)$. Now if in eR, $N = g(eR) \cap eE \neq 0$, then, as $Ne = 0$, we must have $N(eR) = 0$. This is impossible because R is a prime ring. Hence $N = g(eR) \cap eE = 0$, a contradiction to the fact that eE is essential in eR. This shows that eEe must be an essential right ideal of the ring eRe. We use this to show that eR is nonsingular and hence by Claim 2, R_R is also nonsingular. Let Y be the singular submodule of eR. If $Y \neq 0$, then $Ye = eYe \neq 0$ because otherwise YeR would be zero, which is impossible in a prime ring R. Furthermore, for any nonzero $x \in eYe$ there is an essential right ideal $A \subseteq R$ such that $xA = 0$; hence $x(eAe) = 0$ while eAe is an essential right ideal of eRe. This shows that eYe is contained in the right singular ideal of eRe, a contradiction to (iii), that eRe is a prime right Goldie ring (the primeness of eRe follows from that of R; see the proof of (iii) $\Leftrightarrow$ (iv)). Thus $Y = 0$, i.e., eR_R is nonsingular, and so is R_R (see the proof of Claim 2).

By (iii), eRe is a direct sum of, say, n (with $1 < n < \infty$) uniform right ideals. Then (as shown in Claim 1) $e = e_1 + e_2 + \cdots + e_n$ where the e_i's are orthogonal idempotents and each $e_i(eRe)$ is a uniform right ideal of eRe (for convenience we use the same e_i's as before). It follows that $eR = e_1 R \oplus e_2 R \oplus \cdots \oplus e_n R$. For the ring $Q^r_{\max}$, $eQ^r_{\max}e$ is the classical right quotient ring of eRe. Since $e_i(eRe)$ is uniform, $e_i(eQ^r_{\max}e)$ ($=e_i Q^r_{\max}e$) is a minimal right ideal of $eQ^r_{\max}e$. Therefore, $e_i(Q^r_{\max}e)e_i$ ($=e_i Q^r_{\max}e_i$) is a division ring. This in turn implies that $e_i Q^r_{\max}$ is a minimal right ideal of $Q^r_{\max}$ (notice that $Q^r_{\max}$ is again a prime ring). But $e_i Q^r_{\max}$ is the injective hull of $e_i R_R$. Hence $e_i R$ ($i = 1, 2, \dots, n$) is a uniform submodule of eR; in particular this shows that eR_R has finite uniform dimension that is greater than 1.

Let V be a closed uniform submodule of eR; then Ve ($=eVe$) is a uniform right ideal of eRe. Let T be the closure of eVe in the module eRe_{eRe}. Then by (iii), there is an idempotent $g \in eRe$ such that $T = g(eRe)$. As $g(eQ^r_{\max}e)$ is a minimal right ideal of $eQ^r_{\max}e$, $g(eQ^r_{\max}e)g$ ($=gQ^r_{\max}g$) is a division ring (notice that $g = ge = eg$). This shows that $gQ^r_{\max}$ is a minimal right ideal of $Q^r_{\max}$ that is the injective hull of gR. Thus gR is a uniform closed submodule of eR. But $gR \cap V \neq 0$ and V is a uniform closed submodule of eR, too. Hence $V = gR$ because eR_R is nonsingular as shown before. This implies that in eRe, $T^* = eVe$, i.e., eVe is a closed uniform right ideal of eRe.

By (iii), there exists an idempotent $f \in eRe$ such that $eVe = f(eRe)$; in particular, $f \in Ve \subseteq V$. Therefore $fR \subseteq V$. As V is uniform we must have $V = fR$. It follows that V is a direct summand of eR. Now let U be any closed submodule of eR. Since the uniform dimension of U is finite, we can inductively prove that U is a direct summand of eR, i.e., eR_R is CS.

The implication (ii) $\Rightarrow$ (iv) can be proved by an argument similar to that of (i) $\Rightarrow$ (iii); the proof of (iv) $\Rightarrow$ (ii) is similar to the proof of (iii) $\Rightarrow$ (i).

(iii) $\Leftrightarrow$ (iv). This holds by Theorem 1, because it is easy to show that eRe is a prime ring. Namely, if there are two nonzero ideals A and B of eRe that satisfy $AB = 0$, then $(RAR)(RBR) = (RAeR)(ReBR) = RA(eRe)BR = (RA)(BR) = 0$, a contradiction to the primeness of R. Thus eRe is a prime ring. □

Remark. In the proof we did not use the maximal left quotient ring $Q^l_{\max}$ of R because Theorem 1 helps to get the equivalence (iii) $\Leftrightarrow$ (iv). However, if we directly prove (i) $\Rightarrow$ (ii), $Q^l_{\max}$ will play a crucial role in converting the properties of eR_R to ${}_RRe$.

3. Some examples

We present the following examples to clarify some questions related to Theorem 3.

3.1

There are prime right CS nonsingular prime rings that are not left CS. This example shows that the condition u-dim$(eR_R) < \infty$ cannot be removed.

Proof. To see this, we adapt a consideration in [3]. Let V be an infinite-dimensional vector space over a field F, set $Q = End_F(V)$ and $M = \{f \in Q : \dim_F(fV) < \dim_F(V)\}$. Then M is the largest proper two-sided ideal of Q and so Q/M is a simple regular ring. Next let R be the maximal right quotient ring of Q/M. Then R is a simple, von Neumann regular, right self-injective ring. However, as noted in [7, 10.11], R is not directly finite. Consequently, by [7, 9.29], R is not left self-injective. Hence by [7, 13.20], R is not left CS. □

Notice that this ring R does not contain a uniform right (left) ideal, for if it has a uniform right (left) ideal, then R is right (left) Goldie, and hence it is a simple artinian ring, a contradiction. A ring of this type with uniform one-sided ideals will be considered in Example 3.2 below.

3.2

There is a prime right CS ring R containing uniform right ideals and u-dim$(R_R) > 1$. But R is not left CS. Again this example shows that we cannot remove the finiteness of the uniform dimension of eR_R to get the CS property for ${}_RRe$ even for the case where each nonzero right ideal of R contains a minimal right ideal.

Proof. Let K be a division ring and V be a vector space over K with $V = \oplus_{i=1}^{\infty} V_i$ where each V_i is a simple K-module. Let $R = End_K(V)$. Then R is a right self-injective von Neumann regular ring but R is not left self-injective (see [11, XV 3.7]). Clearly the right socle of R is not zero, and u-dim$(R_R) > 1$. Moreover, R is a prime ring, and hence indecomposable as a ring. By [7, 13.20], R is not left CS. □

3.3

The prime ring R in the proof of 3.2 is an example showing that the maximal right and left quotient rings of a prime ring satisfying one of the equivalent conditions in Theorem 3 (for each idempotent $e \in R$ with u-dim$(eR) < \infty$) can be different: As R is right self-injective, but not left self-injective, we have $R = Q^r_{\max}(R)$ but $R \neq Q^l_{\max}(R)$. □

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